AXIALLY SYMMETRIC FLOW IN A THIN LAYER OF FLUID ON THE SURFACE OF A ROTATING SOLID OF REVOLUTION

N. S. Khapilova

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The equations describing the flow of a thin fluid layer on the surface of a rotating solid of revolution in a fixed set of coordinates attached to the body were discussed in [1]. Retaining the notation of [1], we shall confine our attention in the present note to axially symmetric flow.

Analysis in Eqs. (2.3), (2.9), and (2.10) of [1], written in the characteristic form, shows that in the case of a nonstationary axially symmetric flow in a tube of finite length it is necessary to satisfy two boundary conditions on the left, and one such condition on the right, if the flow is subcritical, i.e., $v_1 < (fh)^{1/2}$, or three boundary conditions on the left if the flow is supercritical, i.e. $v_1 > (fh)^{1/2}$.

It is interesting to investigate the possible shapes of the free surface for a given choice of the boundary conditions in a stationary axially symmetric flow.

Let us set $j = 0$ and $q = 0$ in Eqs. (2.3), (2.9), and (2.10), and introduce the new variable $Q = v_1 h$. If we then integrate the equation of continuity, we obtain $Q = c/R(x)$, where c = const. Equation (2.9) then assumes the form

$$\frac{dh}{dx} = \left(\frac{R'}{\sqrt{1-R'^2}} f - \frac{QQ'}{h^2} - \frac{\lambda}{8h^2} vQ\right)\left(f - \frac{Q^2}{h^3}\right)^{-1}. \qquad (1)$$

We now introduce the idea of the critical depth, defined by the condition $fh_k^3 = Q^2 = c^2/R^2$ for given Q. When $h > h_k$ it is supercritical.

Let us also introduce the quantity

$$f_1 = \frac{f}{\sqrt{1-R'^2}} = R\left(\omega + \frac{v_2}{R}\right)^2 > 0 .$$

The depth h_n satisfying the equation

$$\left(f_1 + \frac{c^2}{R^3 h_n^2}\right) R' - \frac{\lambda}{8h_n^2}\frac{c}{R}\left(v_2^2 + \frac{c^2}{R^2 h_n^2}\right)^{1/2} = 0 \qquad (2)$$

will be called the "normal depth."

When $\lambda = 24/\mathrm{Re}$, which corresponds to laminar flow in the layer, we can write Eq. (2) in the form

$$f_1 h_n^3 + \frac{c^2}{R^3} h_n - \frac{3vc}{R'R} = 0 . \qquad (3)$$

This equation has one real and two imaginary roots.

We note that the concept of the normal depth is meaningful only for $R' > 0$, since for $R' < 0$ Eq. (3) has one real root which is always negative. Consider the function

$$\Phi(h) = f_1 h^3 + \frac{c^2}{R^3} h - \frac{3vc}{R'R} .$$

Since $\Phi'(h) > 0$, it follows that $\Phi(h)$ is a monotonically increasing function of h. Moreover, since $\Phi(h_n) = 0$, it follows that $\Phi(h) > 0$ for $h > h_n$, and $\Phi(h) < 0$ for $h < h_n$. The function $\Phi_1(h) = fh^3 - Q^2$ vanishes at the critical depth h_k and is also a monotonically increasing function of h. Consequently, $\Phi_1(h) > 0$ for $h > h_k$, and $\Phi_1(h) < 0$ for $h < h_k$.

In the case of turbulent flows, the coefficient λ can be determined from a formula such as, for example,

$$\psi(h) = F(h, \lambda), \quad \psi(\lambda) = \left(\frac{8}{\lambda}\right)^{1/2},$$
$$F(h, \lambda) = -\frac{20}{\sqrt{g}} \lg\left(\frac{\varepsilon}{h} - \frac{0.385}{\sqrt{1/8\lambda}\mathrm{Re}}\right), \qquad (4)$$

which can be obtained by elementary transformation of the function given in [2]. Let us substitute λ from Eq. (4) into Eq. (2). Since λ is a positive function, we may conclude from the form of Eq. (2) that this equation has no real positive roots when $R' > 0$. From the form of the derivative

$$\Phi_2(h) = f_1 h^2 + \frac{c^2}{R^3} - \frac{\lambda c}{8RR'}\left(\frac{c^2}{R^2h^2} + v_2^2\right)^{1/2},$$

we conclude that it is also a monotonic function of h. Equation (1) can then be written in the form

$$dh/dx = \Phi/\Phi_1 \text{ laminar flow,}$$

$$dh/dx = \Phi_2/\Phi_1 \text{ turbulent flow.}$$

Let $R' > 0$ and $h_n > h_k$. The flow can then occur for depths h such that

$$\text{(a) } h > h_n > h_k, \quad \text{(b) } h_n > h > h_k, \quad \text{(c) } h_k > h .$$

In case (a) it is clear from Eq. (1) that $dh/dx > 0$, i.e., the depth increases during the flow. From Eq. (1) we have

$$\frac{dh}{dx} \to \frac{R'}{\sqrt{1-R'^2}} \quad \text{as} \quad h \to \infty .$$

In case (b) the depth continuously decreases, and $dh/dx \to -\infty$ for $h \to h_k$.

In case (c) the derivative dh/dx is again positive and, moreover,

$$dh/dx \to \infty \quad \text{as} \quad h \to h_k .$$

In the case of laminar flow

$$dh/dx \to 3vR/c \quad \text{as} \quad h \to 0 .$$

The three possible forms of the free surface for $R' > 0$ and $h_n < h_k$ can be obtained in a similar way.

When $R' = 0$, Eq. (1) can be rewritten in the form

$$\frac{dh}{dx} = -\frac{\lambda v v_1}{8h}\left(f - \frac{Q^2}{h^3}\right).$$

For depths greater than the critical value, the flow is such that the depth continuously decreases, and

$$dh/dx \to -\infty \quad \text{as} \quad h \to h_k .$$

For depths less than the critical values, the flow is such that its depth increases.

For $R' < 0$, there are two possible forms of the free surface which are analogous to the case $R' = 0$. In fact, Eq. (1) can be written in the form

$$\frac{dh}{dx} = -\frac{d_1}{f - Q^2/h^2},$$

where

$$d_1 = \left(f_1 + \frac{c^2}{R^3h^2}\right) R_1' + \frac{\lambda}{8h} v v_1, \quad R_1' = -R',$$

is a function which is positive everywhere.

REFERENCES

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